STEP MATHEMATICS 3

2019

Mark Scheme

1. (i) $\dot{x} = -x - y$

So $y = -\dot{x} - x$

As $\dot{y} = x - y$, $-\ddot{x} - \dot{x} = x + \dot{x} + x$

$$\ddot{x} + 2\dot{x} + 2x = 0$$

M1

AQE $\lambda^2 + 2\lambda + 2 = 0$, $\lambda = -1 \pm i$ so $x = e^{-t}(A\cos t + B\sin t)$ **M1 A1 cao**

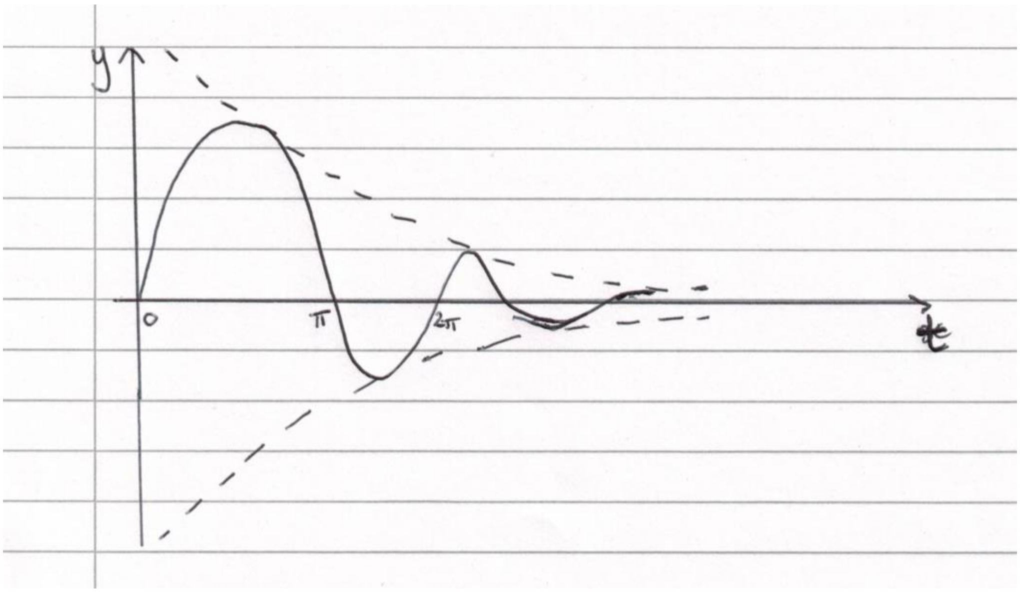
[Alternatively, $x = \dot{y} + y$ leading to $\ddot{y} + 2\dot{y} + 2y = 0$ and $y = e^{-t}(C\cos t + D\sin t)$ etc]

So $y = e^{-t}(A\cos t + B\sin t) - e^{-t}(-A\sin t + B\cos t) - e^{-t}(A\cos t + B\sin t)$

$$= e^{-t}(A\sin t - B\cos t)$$

$t = 0,\ x = 1 \Rightarrow A = 1$ and $t = 0,\ y = 0 \Rightarrow B = 0$

So $x = e^{-t}\cos t$ and $y = e^{-t}\sin t$ **M1 A1 cao**



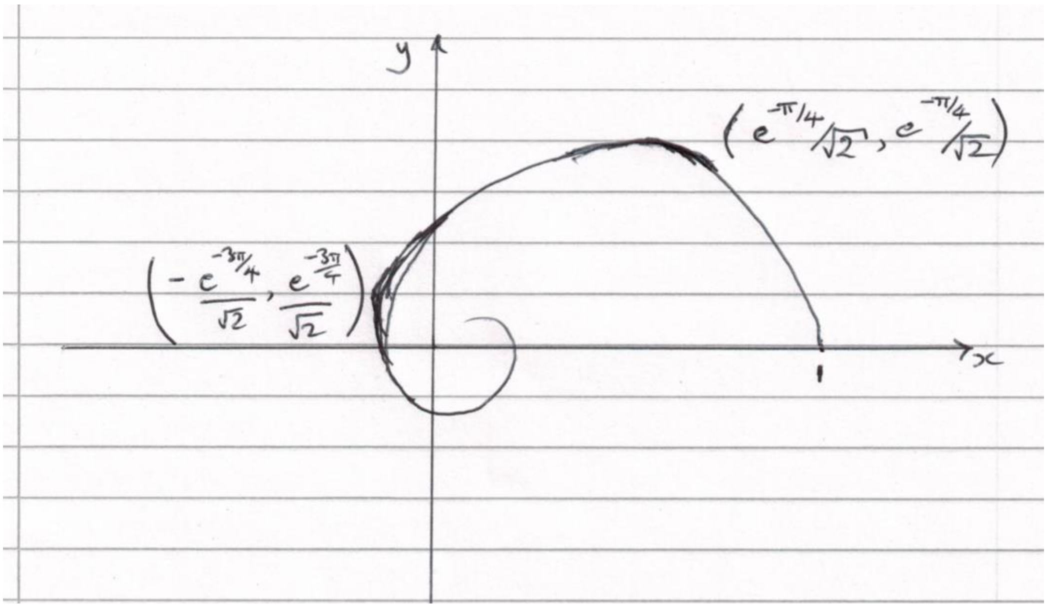
G1 G1 **[7]**

y is greatest when $\dot{y} = 0 \ \Rightarrow x = y \ \Rightarrow \tan t = 1 \ \Rightarrow t = \frac{\pi}{4} + n\pi$

Hence $t = \frac{\pi}{4}$ thus $\left(\frac{e^{-\frac{\pi}{4}}}{\sqrt{2}}, \frac{e^{-\frac{\pi}{4}}}{\sqrt{2}}\right)$ **M1**

x is least when $\dot{x} = 0 \ \Rightarrow x = -y \ \Rightarrow \tan t = -1 \ \Rightarrow t = \frac{3\pi}{4} + n\pi$

Hence $t = \frac{3\pi}{4}$ thus $\left(-\frac{e^{-\frac{3\pi}{4}}}{\sqrt{2}}, \frac{e^{-\frac{3\pi}{4}}}{\sqrt{2}}\right)$ **M1**



G1 G1 G1 **[5]**

(ii) $\dot{x} = -x \quad \Rightarrow x = Ae^{-t}$

$t = 0,\ x = 1 \ \Rightarrow A = 1 \ \Rightarrow x = e^{-t}$ **M1**

So $\dot{y} + y = e^{-t}$

The integrating factor is e^{t} . Thus $e^{t}y = \int 1dt = t + c$

$y = (t + c)e^{-t}$ **M1**

$t = 0,\ y = 0 \ \Rightarrow c = 0$ so $y = te^{-t}$ **A1 cao**

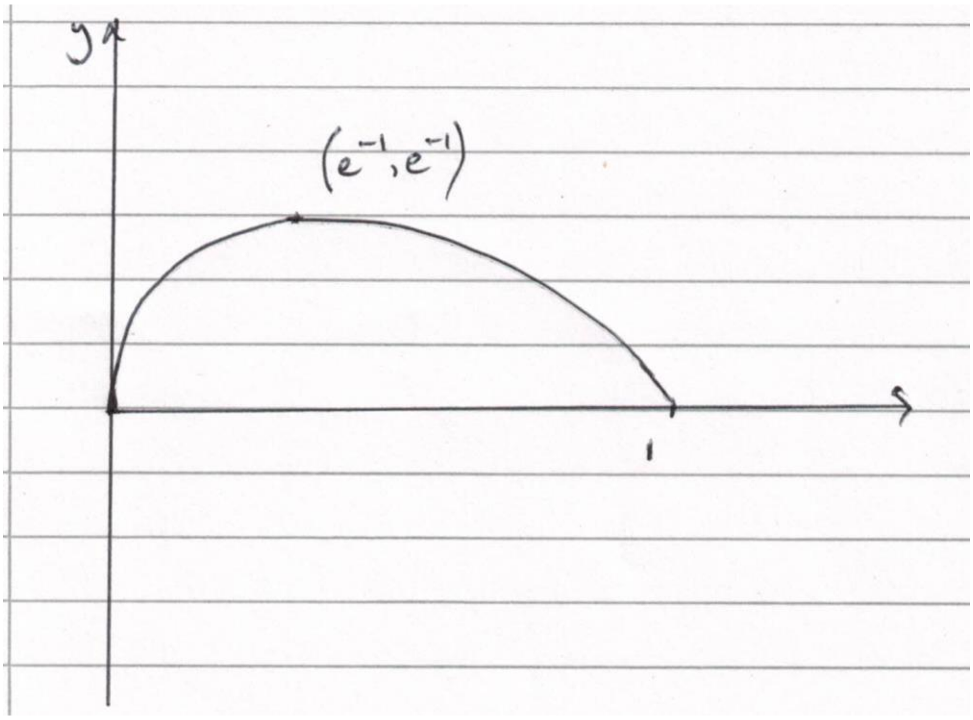
$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{(1-t)e^{-t}}{-e^{-t}} = t - 1$ so there is a maximum at (e^{-1}, e^{-1}) **M1**

As $t \to \infty$, $\frac{dy}{dx} \to \infty$ **M1**

Alternatively,

As $\dot{y} = x - y$, $\ddot{y} = \dot{x} - \dot{y} = -x - \dot{y}$

So $\ddot{y} + 2\dot{y} + y = 0$ yielding $y = (At + B)e^{-t}$ **M1**



G1 G1 G1 **[8]**

2. (i) Let $y = 0$, $f(x+0) = f(x)f(0)\ \forall x$, so $f(x) = f(x)f(0)\ \forall x$ **M1**

Thus $f(x)\big(1-f(0)\big) = 0\ \forall x$. Therefore, $f(x) = 0\ \forall x$, or $f(0) = 1$ **M1**

If $f(x) = 0\ \forall x$ then $f'(x) = 0\ \forall x$ but $f'(0) = k \neq 0$ so $f(0) = 1$ **A1* cso**

$f'(x) = \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right) = \lim_{h\to 0}\left(\frac{f(x)f(h)-f(x)}{h}\right) = f(x)\lim_{h\to 0}\left(\frac{f(h)-1}{h}\right)$ **M1**

But $\lim_{h\to 0}\left(\frac{f(h)-1}{h}\right) = \lim_{h\to 0}\left(\frac{f(0+h)-f(0)}{h}\right) = f'(0) = k$ so $f'(x) = k\,f(x)$ **M1 A1* cso**

Thus

$$\frac{f'(x)}{f(x)} = k$$

Integrating

$$\ln\big(f(x)\big) = kx + c$$

M1

So $f(x) = e^{kx+c} = Ae^{kx}$. As $f(0) = 1$, $A = 1$, so $f(x) = e^{kx}$ **M1 A1 cao [9]**

(ii) Let $y = 0$,

$$g(x+0) = \frac{g(x)+g(0)}{1+g(x)g(0)}\ \forall x$$

M1

Thus $g(x) + \big(g(x)\big)^2 g(0) = g(x) + g(0)\ \forall x$

So $\left(\big(g(x)\big)^2 - 1\right)g(0) = 0\ \forall x$ **M1**

As $|g(x)| < 1$, $\big(g(x)\big)^2 - 1 \neq 0$, and thus $g(0) = 0$ **A1**

$$g'(x) = \lim_{h\to 0}\left(\frac{g(x+h)-g(x)}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{\frac{g(x)+g(h)}{1+g(x)g(h)} - g(x)}{h}\right) = \lim_{h\to 0}\left(\frac{g(x)+g(h)-g(x)\big(1+g(x)g(h)\big)}{h\big(1+g(x)g(h)\big)}\right)$$

M1

$$= \lim_{h\to 0}\left(\frac{g(h)\left(1-\big(g(x)\big)^2\right)}{h\big(1+g(x)g(h)\big)}\right) = \left(1-\big(g(x)\big)^2\right)\lim_{h\to 0}\left(\frac{g(h)}{h\big(1+g(x)g(h)\big)}\right)$$

M1

$$\lim_{h\to 0}\left(\frac{g(h)}{h\big(1+g(x)g(h)\big)}\right) = \lim_{h\to 0}\left(\frac{g(h)/h}{\big(1+g(x)g(h)\big)}\right) = \lim_{h\to 0}\left(\frac{g(h)}{h}\right) = g'(0) = k$$

M1 A1

Thus $g'(x) = k\left(1-\big(g(x)\big)^2\right)$ **A1**

$$\frac{g'(x)}{\left(1-\left(g(x)\right)^2\right)} = k$$

Integrating,

$$\tanh^{-1}\left(g(x)\right) = kx + c$$

M1

$$g(x) = \tanh(kx + c)$$

As $g(0) = 0$, $\tanh(c) = 0$ and so $c = 0$ **M1**

Thus

$$g(x) = \tanh(kx)$$

A1 **[11]**

[Alternatively, integrating having used partial fractions,

$$\frac{1}{2}\ln\left(\frac{1+g(x)}{1-g(x)}\right) = kx + c$$

M1

As $g(0) = 0$, $c = 0$

$$\frac{1+g(x)}{1-g(x)} = e^{2kx}$$

M1

and so

$$g(x) = \frac{e^{2kx}-1}{e^{2kx}+1}$$

A1 **[3]**

3. (i) $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow ax + by = x \quad cx + dy = y$ **M1**

$(a-1)x = -by$

$-cx = (d-1)y$

Thus $(a-1)x(d-1)y = bycx$, that is $(a-1)(d-1)xy - bcxy = 0$

So $\big((a-1)(d-1) - bc\big)xy = 0$ **M1**

Thus $\big((a-1)(d-1) - bc\big) = 0$ or $x = 0$ or $y = 0$ **M1**

If L_1 is $x = 0$, then both $by = 0$ and $dy = y$, $\forall y$

Thus, $b = 0$ and $d = 1$, meaning that $\big((a-1)(d-1) - bc\big) = 0$ **E1**

Similarly, if L_1 is $y = 0$, then both $cx = 0$ and $ax = x$, $\forall x$

Then $c = 0$ and $a = 1$, meaning that $\big((a-1)(d-1) - bc\big) = 0$

In all three cases, $(a-1)(d-1) = bc$ **E1**

Alternatively,

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ **M1**

So $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

That is $\begin{pmatrix} a-1 & b \\ c & d-1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ **M1**

As this is true for a line of invariant points, it does not have a unique solution and so **E1**

$det\begin{pmatrix} a-1 & b \\ c & d-1 \end{pmatrix} = 0$ **M1**

and so $\big((a-1)(d-1) - bc\big) = 0$ which implies both $\big((a-1)(d-1) - bc\big)xy = 0$ and $(a-1)(d-1) = bc$ **E1**

If L_1 does not pass through the origin then either L_1 is a) $y = mx + k$ with $k \neq 0$

or b) $x = k$ with $k \neq 0$ **E1**

For a) $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ mx+k \end{pmatrix} = \begin{pmatrix} x \\ mx+k \end{pmatrix} \quad \forall x$

Thus $ax + b(mx+k) = x$ and $cx + d(mx+k) = mx + k$

As these apply for all x, and $k \neq 0$, $bk = 0$ which implies $b = 0$ and $a + bm = 1$ and thus $a = 1$

Also $dk = k$ implying $d = 1$ and $c + dm = m$ which thus gives $c = 0$ **M1**

For b) $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} k \\ y \end{pmatrix} = \begin{pmatrix} k \\ y \end{pmatrix} \ \forall y$

Thus $ak + by = k$ and $ck + dy = y$

So $ak = k$ implying $a = 1$ and $b = 0$ and $ck = 0$ implying $c = 0$ and $d = 1$ **M1**

Thus $\boldsymbol{A} = \boldsymbol{I}$ **A1** **[9]**

(ii) If $(a-1)(d-1) = bc$ and $b \neq 0$

then $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$ is an invariant point iff $ax + by = x$ and $cx + dy = y$ **M1**

That is $(a-1)x + by = 0$, $(a-1)(d-1)x + b(d-1)y = 0$, $bcx + b(d-1)y = 0$ and so $cx + (d-1)y = 0$ as required. **E1**

The line of invariant points is thus $(a-1)x + by = 0$ which is $cx + (d-1)y = 0$ **A1**

If $(a-1)(d-1) = bc$ and $b = 0$ then $a = 1$ or if $a \neq 1$, $d = 1$

$a = 1$, $\begin{pmatrix} 1 & 0 \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ cx + dy \end{pmatrix}$ so points on $cx + (d-1)y = 0$ are invariant. **B1**

$a \neq 1$, $\begin{pmatrix} a & 0 \\ c & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax \\ cx + y \end{pmatrix}$ so points on $x = 0$ are invariant. **B1** **[5]**

(iii) L_2 is an invariant line implies $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ mx + k \end{pmatrix} = \begin{pmatrix} x' \\ mx' + k \end{pmatrix}$ and as L_2 does not pass through the origin $k \neq 0$ **M1**

So $ax + b(mx + k) = x'$ and $cx + d(mx + k) = mx' + k$

As these are true for all x, true for $x = 0$ and thus $bk = x'$ and $dk = mx' + k$ giving

$dk = mbk + k$ and as $k \neq 0$, $d = mb + 1$ **E1**

Similarly, for $x = 1$ and thus $a + b(m + k) = x''$ and $c + d(m + k) = mx'' + k$

So $c + d(m + k) = m\big(a + b(m + k)\big) + k$

$c + dm + dk = ma + (m + k)(d - 1) + k$

$c + dm + dk = ma + dm + dk - m - k + k$

$c = ma - m$ **E1**

So $m(a-1) = c$ and $d - 1 = mb$

Hence, multiplying these $m(a-1)(d-1) = mbc$ **E1**

Thus, if $m \neq 0$, $(a-1)(d-1) = bc$

If $m = 0$, $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ k \end{pmatrix} = \begin{pmatrix} x' \\ k \end{pmatrix}$ giving $ax + bk = x'$ and $cx + dk = k$ **E1**

As these must be true for all x, when $x = 0$, $dk = k$ giving $d = 1$ and so for choosing any $x \neq 0$, we find $c = 0$. Thus $(a-1)(d-1) = 0$ and $bc = 0$ giving $(a-1)(d-1) = bc$

E1 [6]

4. (i) Degree 1, $x - a_1 = 0$ has root $x = a_1$ and so $x - a_1$ is reflexive. **B1**

Degree 2, $x^2 - a_1 x + a_2 = 0$ has to have roots a_1 and a_2 to be reflexive.

Thus $a_1{}^2 - a_1 a_1 + a_2 = 0$ (giving $a_2 = 0$) and $a_2{}^2 - a_1 a_2 + a_2 = 0$ **M1** which is consistent. Thus, $x^2 - a_1 x\ (+\ 0)$ **B1**

Alternatively

$a_1 = a_1 + a_2$ and $a_2 = a_1 a_2$ giving $a_2 = 0$ and consistent for any a_1 **M1**

Thus, $x^2 - a_1 x\ (+\ 0)$ **B1**

Degree 3, $x^3 - a_1 x^2 + a_2 x - a_3 = 0$

$a_1 = a_1 + a_2 + a_3$

$a_2 = a_1 a_2 + a_2 a_3 + a_3 a_1$

$a_3 = a_1 a_2 a_3$ **M1**

The first equation implies that $a_2 + a_3 = 0$ **M1** or in other words $a_2 = -a_3$

This result substituted into the second equation implies that $a_2 = a_2 a_3$ **M1**

Continuing $a_2 a_3 - a_2 = 0$, $a_2(a_3 - 1) = 0$ **M1** so either $a_2 = 0$ and thus $a_3 = 0$ in which case the equations are consistent for any a_1 **M1** or $a_3 = 1$ and thus $a_2 = -1$ and $a_1 = -1$ from the third equation. **M1**

Alternatively, from the third equation, $a_3 = a_1 a_2 a_3$, $a_1 a_2 a_3 - a_3 = 0$, $a_3(a_1 a_2 - 1) = 0$

so $a_3 = 0$, $a_2 = 0$ and consistent for any a_1 or $a_1 a_2 = 1$ In the latter case it is simpler to use equation two route again.

Yielding $x^3 - a_1 x^2$ **A1** or $x^3 + x^2 - x - 1$ **A1**

Alternative approach $a_1{}^3 - a_1 a_1{}^2 + a_2 a_1 - a_3 = 0$ (A), $a_2{}^3 - a_1 a_2{}^2 + a_2 a_2 - a_3 = 0$ (B),and

$a_3{}^3 - a_1 a_3{}^2 + a_2 a_3 - a_3 = 0$ (C) **M1**

So (A) implies $a_3 = a_1 a_2$ and thus (B) becomes $a_2{}^3 - a_1 a_2{}^2 + a_2{}^2 - a_1 a_2 = 0$

Hence $a_2(a_2{}^2 - a_1 a_2 + a_2 - a_1) = a_2(a_2 + 1)(a_2 - a_1) = 0$ **M1**

Therefore, $a_2 = 0$ and so from (A) $a_3 = 0$ giving the polynomial $x^3 - a_1 x^2$, which is reflexive **A1**

or $a_2 = -1$ and so from (A) $a_3 = -a_1$ giving the polynomial

$x^3 - a_1x^2 \; - x + \; a_1 = (x - \; a_1)(x^2 - 1)$ for which the equation has roots a_1 and ± 1 . Thus $a_3 = -a_1 = 1$ so the polynomial is $x^3 + x^2 \; - x - 1$ for which the equation has roots -1 , -1 and 1 and so is reflexive **M1 A1**

or $a_2 = a_1$ and so A gives $a_3 = {a_1}^2$ But C is ${a_3}^3 - a_1{a_3}^2 \; + \; a_2a_3 \; - a_3 = 0$ and so

$a_3({a_3}^2 - a_1a_3 \; + \; a_1 \; - a_3) = 0$, $a_3(a_3 \; - a_1)(a_3 - 1) = 0$ **M1**

$a_3 = 0$, $a_1 = 0$, $a_2 = 0$ giving the reflexive polynomial x^3

$a_3 = a_1$, implies $a_1 = 0$ or $a_1 = 1$ giving x^3 again or $a_1 = a_2 = a_3 = 1$ giving the polynomial $x^3 - x^2 \; + \; x \; - 1 = (\, x^2 + 1)(\, x \; - 1)$ which is not reflexive. **E1**

$a_3 = 1$, implies $a_1 = a_2 = \pm 1$ giving 1,1,1 which is not possible or -1,-1,1 (both already considered) **E1** **[11]**

Summary degree 1 $x - a_1$, degree 2 $x^2 - a_1x \;\; (+ \;\; 0)$, and degree 3 $x^3 - a_1x^2$ or

` $x^3 + x^2 \; - x - 1$

(ii)

$$a_1 = \sum_{r=1}^{n} a_r$$

and so,

$$\sum_{r=2}^{n} a_r = 0$$

B1

$$a_2 = \frac{1}{2}\sum_{i \neq j} a_i a_j$$

$$\left(\sum_{r=2}^{n} a_r\right)^2 = \sum_{r=2}^{n} {a_r}^2 + \sum_{i \neq j} a_i a_j - 2a_1 \sum_{r=2}^{n} a_r$$

M1

Thus

$$0 = \sum_{r=2}^{n} {a_r}^2 + 2a_2$$

Hence

$$2a_2 = -{a_2}^2 - \; {a_3}^2 - \; \cdots - {a_n}^2$$

as required. **A1***

$${a_2}^2 + 2a_2 + 1 = 1 - \; {a_3}^2 - \; \cdots - {a_n}^2$$

Thus,

$$1 - a_3{}^2 - \cdots - a_n{}^2 \geq 0$$

So

$$a_3{}^2 + \cdots + a_n{}^2 \leq 1$$

M1

If all the coefficients are integers, then as $a_n \neq 0$, $a_n{}^2 = 1$ and the other coefficients for $r = 3, \ldots, n-1$ are zero. But

$$a_n = \prod_{r=1}^{n} a_r$$

so we have established a contradiction. Thus if $a_n \neq 0$, $n \leq 3$ **E1** **[5]**

(iii) So apart from those found in (i) with a_1 integer, any other reflexive polynomials must have $a_n = 0$ **E1**

So

$$x^n - a_1 x^{n-1} + a_2 x^{n-2} - \cdots + (-1)^{n-1} a_{n-1} x = 0$$

$$\therefore \; x(x^{n-1} - a_1 x^{n-2} + a_2 x^{n-3} - \cdots + (-1)^{n-1} a_{n-1}) = 0$$

M1

which gives a root of zero plus the roots of the bracketed expression. Thus we require the bracketed expression to be itself a reflexive polynomial. This can only happen if either the bracketed expression is of degree 3 or, in turn, $a_{n-1} = 0$, and so on. **E1**

Hence, we have $x - a_1$, $x^2 - a_1 x \; (+\; 0)$, $x^3 - a_1 x^2$, (with a_1 integer),` $x^3 + x^2 - x - 1$, or these multiplied by x^r.

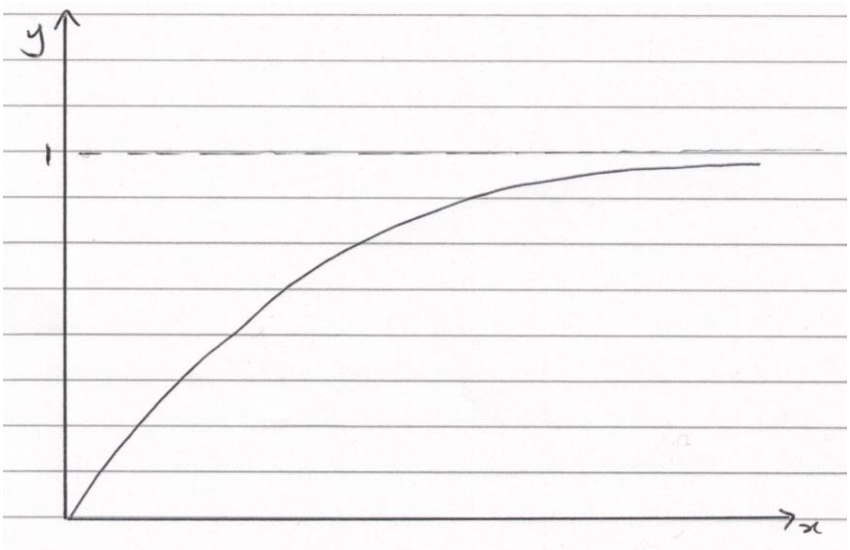
That is $(x - a_1)x^r$ or $(x+1)^2(x-1)x^r$ with $r = 0, 1, 2, \ldots$ **A1** **[4]**

5. (i)

$$f(x) = \frac{x}{\sqrt{x^2+p}}$$

$$f'(x) = \frac{\sqrt{x^2+p} - x\frac{1}{2}\frac{2x}{\sqrt{x^2+p}}}{x^2+p} = \frac{p}{(x^2+p)^{\frac{3}{2}}}$$

M1



G1

G1 **[3]**

(ii) for answers using substitution from question paper

$$y = \frac{bx}{\sqrt{x^2+p}} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{bp}{(x^2+p)^{\frac{3}{2}}}$$

$$b^2 - y^2 = b^2 - \frac{b^2x^2}{x^2+p} = \frac{(b^2-b^2)x^2 + b^2p}{x^2+p} = \frac{b^2p}{x^2+p}$$

M1

$$c^2 - y^2 = c^2 - \frac{b^2x^2}{x^2+p} = \frac{(c^2-b^2)x^2 + c^2p}{x^2+p}$$

M1

So

$$\int \frac{1}{(b^2-y^2)\sqrt{c^2-y^2}}\, dy = \int \frac{(x^2+p)}{b^2p} \frac{\sqrt{x^2+p}}{\sqrt{[(c^2-b^2)x^2+c^2p]}} \frac{bp}{(x^2+p)^{\frac{3}{2}}}\, dx$$

M1

$$= \int \frac{1}{b\sqrt{[(c^2-b^2)x^2+c^2p]}}\, dx$$

M1 [4]

Let $c^2 = 2$ **M1**

Let $b^2 = 3$ **M1**

Thus

$$\int_1^{\sqrt{2}} \frac{1}{(3-y^2)\sqrt{2-y^2}}\, dy = \int_?^? \frac{1}{3+x^2}\, dx = \left[\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right)\right]_?^?$$

M1 **M1** **M1** **[5]**

Let $y = \frac{1}{x}$, $\frac{dy}{dx} = -\frac{1}{x^2}$ **M1**

$$\int_{\frac{1}{\sqrt{2}}}^{1} \frac{y}{(3y^2-1)\sqrt{2y^2-1}}\, dy = \int_{\sqrt{2}}^{1} \frac{1}{x}\frac{1}{\frac{3}{x^2}-1}\frac{1}{\sqrt{\frac{2}{x^2}-1}} \times -\frac{1}{x^2}\, dx = \int_1^{\sqrt{2}} \frac{1}{(3-x^2)\sqrt{2-x^2}}\, dx$$

M1

and so is same answer as previous part **A1 ft** **[3]**

(ii) for answers using correct substitution where candidates have realised there is a misprint

$$y = \frac{cx}{\sqrt{x^2+p}} \Rightarrow \frac{dy}{dx} = \frac{cp}{(x^2+p)^{\frac{3}{2}}}$$

$$b^2 - y^2 = b^2 - \frac{c^2x^2}{x^2+p} = \frac{(b^2-c^2)x^2 + b^2p}{x^2+p}$$

$$c^2 - y^2 = c^2 - \frac{c^2x^2}{x^2+p} = \frac{c^2p}{x^2+p}$$

M1

Choose $p = 1$ **B1**

So

$$\int \frac{1}{(b^2-y^2)\sqrt{c^2-y^2}}\, dy = \int \frac{x^2+1}{b^2+(b^2-c^2)x^2}\frac{\sqrt{x^2+1}}{c}\frac{c}{(x^2+1)^{\frac{3}{2}}}\, dx$$

M1

$$= \int \frac{1}{b^2+(b^2-c^2)x^2}\, dx$$

A1* **[4]**

Let $c^2 = 2$ Then $(x^2+1)y^2 = 2x^2$ so $x^2 = \frac{y^2}{2-y^2}$ and when $y = 1, x = 1$

and $y \to \sqrt{2}, x \to \infty$

Let $b^2 = 3$ **B1 M1 A1**

Thus

$$\int_1^{\sqrt{2}} \frac{1}{(3-y^2)\sqrt{2-y^2}}\, dy = \int_1^{\infty} \frac{1}{3+x^2} dx = \left[\frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{x}{\sqrt{3}}\right)\right]_1^{\infty} = \frac{1}{\sqrt{3}}\left(\frac{\pi}{2} - \frac{\pi}{6}\right) = \frac{\pi}{3\sqrt{3}}$$

M1 A1 **{5]**

Let $y = \frac{1}{x}$, $\frac{dy}{dx} = -\frac{1}{x^2}$ **M1**

$$\int_{\frac{1}{\sqrt{2}}}^{1} \frac{y}{(3y^2-1)\sqrt{2y^2-1}}\, dy = \int_{\sqrt{2}}^{1} \frac{1}{x} \frac{1}{\frac{3}{x^2}-1} \frac{1}{\sqrt{\frac{2}{x^2}-1}} \times -\frac{1}{x^2} dx = \int_1^{\sqrt{2}} \frac{1}{(3-x^2)\sqrt{2-x^2}}\, dx$$

M1

and so is $\frac{\pi}{3\sqrt{3}}$ **A1 ft** **[3]**

(iii) If

$$y = \frac{bx}{\sqrt{x^2+p}} \Rightarrow \frac{dy}{dx} = \frac{bp}{(x^2+p)^{\frac{3}{2}}}$$

and so

$(x^2+p)y^2 = b^2x^2$, $x^2 = \frac{py^2}{b^2-y^2}$ **M1**

thus

$$\int \frac{1}{(3y^2-1)\sqrt{2y^2-1}}\, dy = \int \frac{1}{\left(\frac{3b^2x^2}{x^2+p} - 1\right)\sqrt{\frac{2b^2x^2}{x^2+p} - 1}} \frac{bp}{(x^2+p)^{\frac{3}{2}}}\, dx$$

$$= \int \frac{bp}{(3b^2x^2 - (x^2+p))} \frac{1}{\sqrt{2b^2x^2 - (x^2+p)}}\, dx$$

M1

Choosing $2b^2 = 1$ and $p = -1$ we have **B1**

$$\int_{\frac{1}{\sqrt{2}}}^{1} \frac{1}{(3y^2-1)\sqrt{2y^2-1}}\, dy = \int_{\infty}^{\sqrt{2}} \frac{-\frac{1}{\sqrt{2}}}{\frac{1}{2}x^2+1}\, dx = \sqrt{2}\int_{\sqrt{2}}^{\infty} \frac{1}{x^2+2}\, dx$$

M1

$$= \sqrt{2}\left[\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right)\right]_{\sqrt{2}}^{\infty} = \frac{\pi}{4}$$

A1 [5]

6. $|z-a|^2 = (z-a)(z-a)^* = (z-a)(z^*-a^*) = zz^* - az^* - a^*z + aa^*$

M1

So z satisfies $|z-a|^2 = r^2$ which means that the locus of P is a circle (C) centre a and radius r (which does not pass through the origin as $r^2 \neq aa^*$.) **A1** **[2]**

(i) As $w = \frac{1}{z}$, $z = \frac{1}{w}$ so

$$\frac{1}{w}\frac{1}{w^*} - a\frac{1}{w^*} - a^*\frac{1}{w} + aa^* - r^2 = 0$$

M1

$$(aa^* - r^2)ww^* - aw - a^*w^* + 1 = 0$$

$$ww^* - \frac{a}{aa^*-r^2}w - \frac{a^*}{aa^*-r^2}w^* + \frac{1}{aa^*-r^2} = 0$$

$$ww^* - \left(\frac{a^*}{aa^*-r^2}\right)^* w - \left(\frac{a^*}{aa^*-r^2}\right)w^* + \left(\frac{a^*}{aa^*-r^2}\right)\left(\frac{a}{aa^*-r^2}\right)$$
$$= \left(\frac{a^*}{aa^*-r^2}\right)\left(\frac{a}{aa^*-r^2}\right) - \frac{1}{aa^*-r^2}$$

$$\left|w - \frac{a^*}{aa^*-r^2}\right|^2 = \frac{aa^*}{(aa^*-r^2)^2} - \frac{1}{aa^*-r^2} = \frac{r^2}{(aa^*-r^2)^2}$$

M1

So C' is a circle centre $\frac{a^*}{aa^*-r^2}$ with radius $\left|\frac{r}{aa^*-r^2}\right|$ **A1** **[3]**

If C and C' are the same circle, then

$$a = \frac{a^*}{aa^*-r^2}$$

M1

and

$$r^2 = \left|\frac{r}{aa^*-r^2}\right|^2$$

M1

Thus

$$r^2(|a|^2 - r^2)^2 = r^2$$

and dividing by $r^2 \neq 0$, $(|a|^2 - r^2)^2 = 1$ as required. **A1*** **[3]**

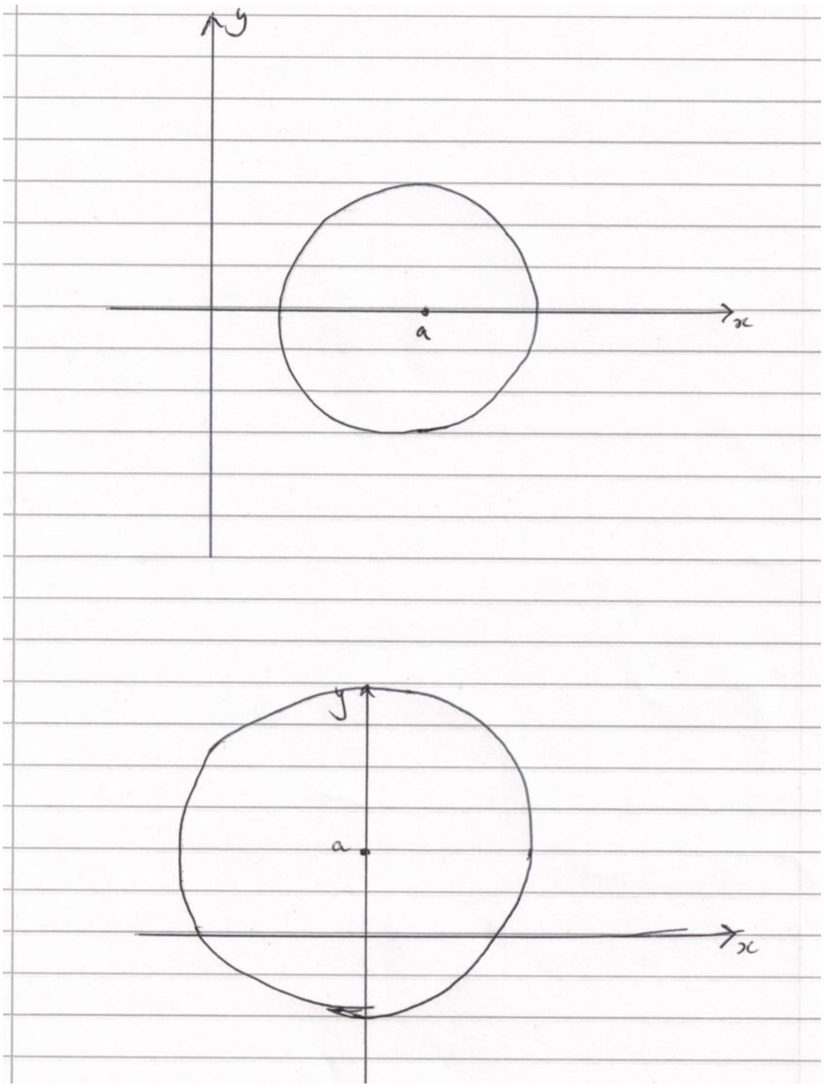
So $|a|^2 - r^2 = \pm 1$ and the equation equating the centres becomes $a = \pm a^*$ **M1**

If $a = c + di$, and $a = a^*$, $c + di = c - di$ implying $d = 0$ and hence a is real. **M1 A1**

G1

If $a = c + di$, and $a = -a^*$, $c + di = -c + di$ implying $c = 0$ and hence a is imaginary. **M1 A1**

G1 **[7]**



(ii) If $w = \frac{1}{z^*}$ then

$$\left|w^* - \frac{a^*}{aa^* - r^2}\right|^2 = \frac{r^2}{(aa^* - r^2)^2}$$

M1

so

$$\left|w - \frac{a}{aa^* - r^2}\right|^2 = \frac{r^2}{(aa^* - r^2)^2}$$

M1

and thus, as before $|a|^2 - r^2 = \pm 1$ but now $a = \frac{a}{aa^* - r^2}$ **A1**

So in the case $|a|^2 - r^2 = +1$, then any a with $|a| = \sqrt{r^2+1}$ is possible; in the case $|a|^2 - r^2 = -1$, $a = 0$ (in which case $r = 1$) **A1**

So, it is not the case that a is either real or imaginary. **A1** **[5]**

7. (i)

$$y^2(y^2-a^2)=x^2(x^2-a^2)$$

$$y^4-a^2y^2+\frac{a^4}{4}=x^4-a^2x^2+\frac{a^4}{4}$$

M1

$$\left(y^2-\frac{a^2}{2}\right)^2=\left(x^2-\frac{a^2}{2}\right)^2$$

$$y^2-\frac{a^2}{2}=\pm\left(x^2-\frac{a^2}{2}\right)$$

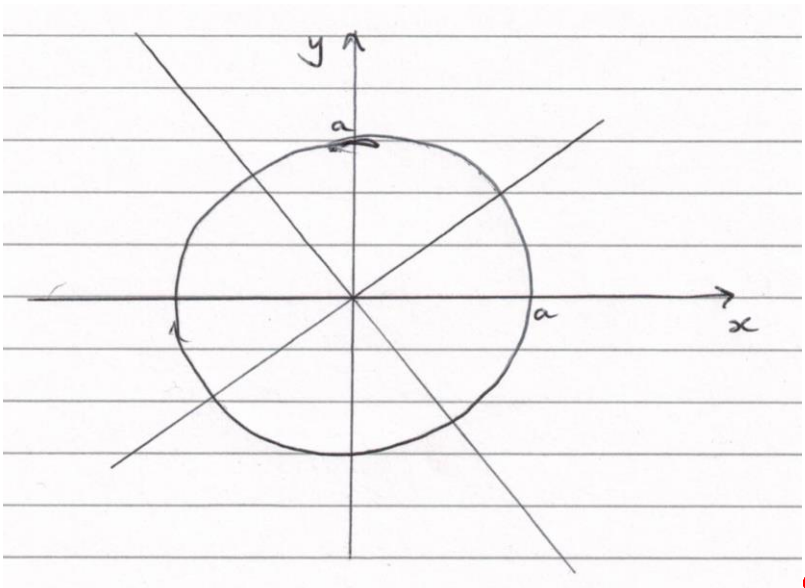
So $x^2=y^2$ which gives $y=\pm x$

or $x^2+y^2=a^2$ **A1**

Alternatively

$$y^4-x^4=a^2y^2-a^2x^2$$

so $(y^2-x^2)(y^2+x^2)-a^2(y^2-x^2)=0$ **M1**



G1

[3]

(ii)

$$y^2(y^2-5)=x^2(x^2-4)$$

a)

$$(x^2)^2-4x^2-y^2(y^2-5)=0$$

So for real x^2,

$$16+4y^2(y^2-5)\geq 0$$

M1

$$(y^2)^2 - 5y^2 + 4 \geq 0$$

$$(y^2-1)(y^2-4) \geq 0$$

$$(y-1)(y+1)(y-2)(y+2) \geq 0$$

M1

As $y \geq 0$,

$$(y-1)(y-2) \geq 0$$

So $0 \leq y \leq 1$ or $y \geq 2$ A1* [3]

b) For small x and y,

$y^4 - 5y^2 = x^4 - 4x^2$ becomes $5y^2 \approx 4x^2$ so $y \approx \pm\frac{2x}{\sqrt{5}}$ B1

For large x and y,

$y^4 - 5y^2 = x^4 - 4x^2$ becomes $y^4 \approx x^4$ so $y \approx \pm x$ B1 [2]

c)

$$y^2(y^2-5) = x^2(x^2-4)$$

$$(4y^3 - 10y)\frac{dy}{dx} = 4x^3 - 8x$$

M1

$$\frac{dy}{dx} = 0 \Rightarrow 4x^3 - 8x = 0$$

$$4x(x^2-2) = 0$$

M1

So $x = 0$, $y = 0,\ \sqrt{5}$ or $x = \sqrt{2}$, $y = 1$, 2

Thus $(0, \sqrt{5})$, $(\sqrt{2}, 1)$, $(\sqrt{2}, 2)$ but not $(0, 0)$ A1 B1

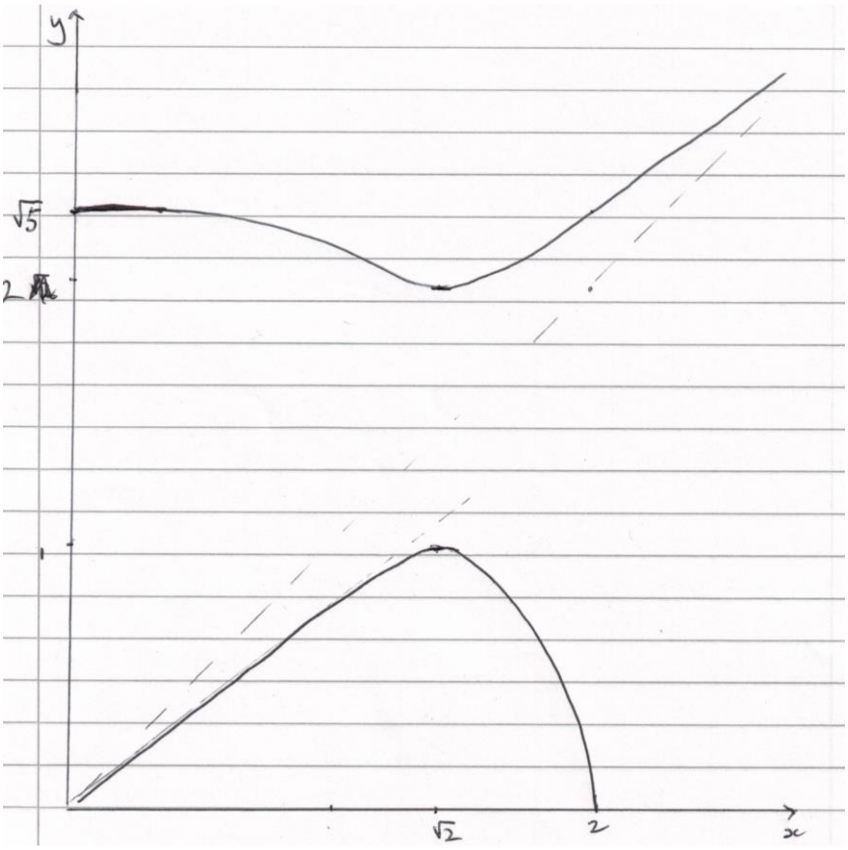
$$\frac{dx}{dy} = 0 \Rightarrow 4y^3 - 10y = 0$$

$$2y(2y^2-5) = 0$$

M1

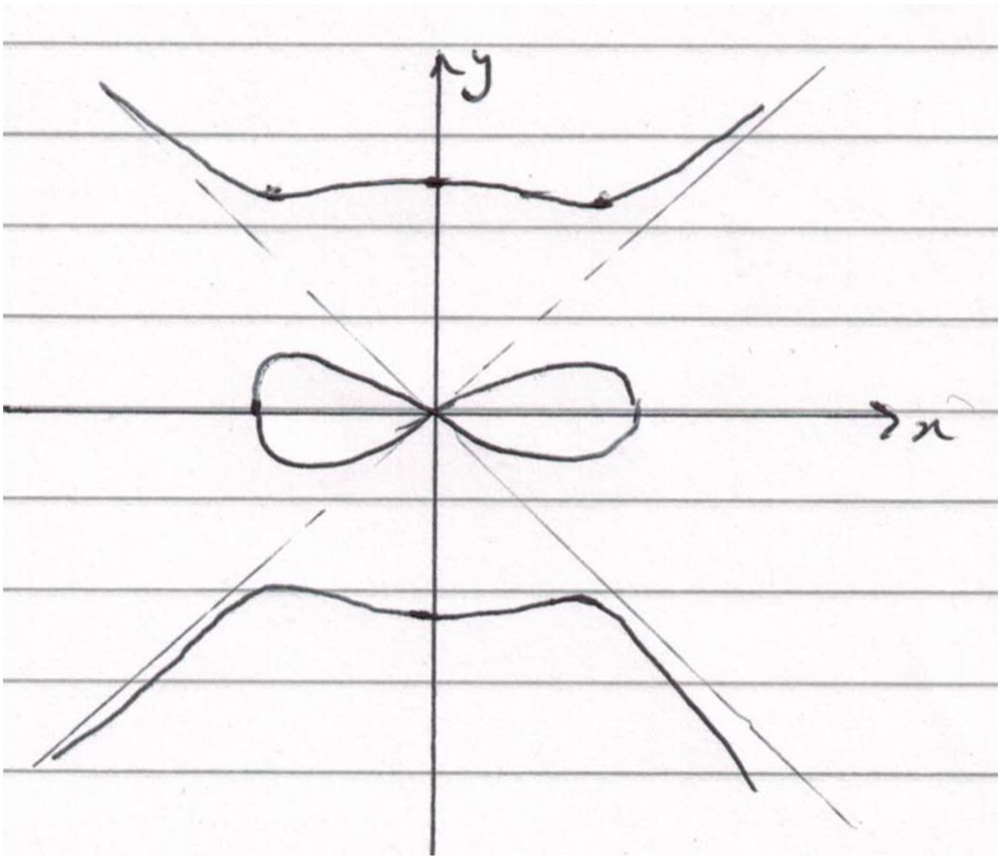
So $y = 0$, $x = 0,\ 2$ but $y = \sqrt{\frac{5}{2}}$ gives x complex E1

Thus $(2, 0)$ but not $(0, 0)$ A1 [7]



G1 G1 G1 G1 **[4]**

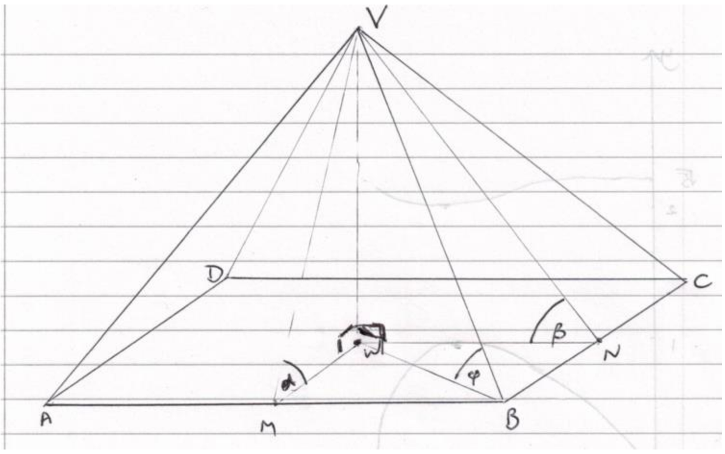
(iii) **G1 ft** **[1]**



8. (i) If W is centre of base, and M is midpoint of AB, then

$$\overrightarrow{MV} = \lambda \begin{pmatrix} 0 \\ \cos\alpha \\ \sin\alpha \end{pmatrix}$$

M1



So a unit vector perpendicular to AVB is $\begin{pmatrix} 0 \\ -\sin\alpha \\ \cos\alpha \end{pmatrix}$ **A1 (or B2)**

Similarly, a unit vector perpendicular to BVC is $\begin{pmatrix} \sin\beta \\ 0 \\ \cos\beta \end{pmatrix}$ **B1**

AS the obtuse angle between AVB and BVC is $\pi - \theta$, the acute angle between the two unit vectors is θ , and so

$$\begin{pmatrix} 0 \\ -\sin\alpha \\ \cos\alpha \end{pmatrix} \cdot \begin{pmatrix} \sin\beta \\ 0 \\ \cos\beta \end{pmatrix} = \cos\theta$$

M1

Hence $\cos\theta = \cos\alpha\cos\beta$ as required. **A1*** **[5]**

(ii) $MW = VW\cot\alpha$, $BM = NW = VW\cot\beta$, $BW = VW\cot\varphi$ **M1**

By Pythagoras, $MW^2 + BW^2 = BW^2$ and so $\cot^2\alpha + \cot^2\beta = \cot^2\varphi$ **M1**

So

$$\tan^2\varphi = \frac{1}{\cot^2\alpha + \cot^2\beta}$$

and thus

$$\sec^2\varphi = \frac{1}{\cot^2\alpha + \cot^2\beta} + 1$$

M1

giving

$$\cos^2\varphi = \frac{\cot^2\alpha + \cot^2\beta}{\cot^2\alpha + \cot^2\beta + 1} = \frac{\tan^2\alpha + \tan^2\beta}{\tan^2\alpha + \tan^2\beta + \tan^2\alpha\tan^2\beta}$$

M1

$$= \frac{\sec^2\alpha + \sec^2\beta - 2}{\sec^2\alpha + \sec^2\beta - 2 + (\sec^2\alpha - 1)(\sec^2\beta - 1)}$$

M1

$$= \frac{\cos^2\alpha + \cos^2\beta - 2\cos^2\alpha\cos^2\beta}{\cos^2\alpha + \cos^2\beta - 2\cos^2\alpha\cos^2\beta + (1-\cos^2\alpha)(1-\cos^2\beta)}$$

M1

$$= \frac{\cos^2\alpha + \cos^2\beta - 2\cos^2\theta}{1 - \cos^2\alpha\cos^2\beta}$$

$$= \frac{\cos^2\alpha + \cos^2\beta - 2\cos^2\theta}{1 - \cos^2\theta}$$

M1 A1* [8]

$$(\cos\alpha - \cos\beta)^2 \geq 0$$

Thus

$$\cos^2\alpha + \cos^2\beta \geq 2\cos\alpha\cos\beta = 2\cos\theta$$

M1

So

$$\cos^2\varphi = \frac{\cos^2\alpha + \cos^2\beta - 2\cos^2\theta}{1 - \cos^2\theta} \geq \frac{2\cos\theta - 2\cos^2\theta}{1 - \cos^2\theta}$$

A1* **[2]**

$$\frac{2\cos\theta - 2\cos^2\theta}{1 - \cos^2\theta} = \frac{2\cos\theta\,(1 - \cos\theta)}{(1 - \cos\theta)(1 + \cos\theta)}$$

$1 - \cos\theta \neq 0$ as θ is acute. **E1**

Thus

$$\cos^2\varphi \geq \frac{2\cos\theta}{(1 + \cos\theta)} = \frac{2}{(1 + \cos\theta)}\cos\theta$$

As θ is acute, $(1 + \cos\theta) < 2$ and so $\frac{2}{(1+\cos\theta)}\cos\theta > \cos\theta$ **E1**

But also $\cos\theta > \cos\theta\cos\theta$ **E1**

Hence, $\cos^2\varphi \geq \cos^2\theta$

As both $\cos\varphi$ and $\cos\theta$ are positive $\cos\varphi \geq \cos\theta$, and so $\varphi \leq \theta$ **E1 E1** **[5]**

9. (i)

$$\boldsymbol{r} = (a\sin\theta - s)\boldsymbol{i} + a\cos\theta\,\boldsymbol{j}$$

B1

So differentiating with respect to time,

$$\dot{\boldsymbol{r}} = \left(a\dot{\theta}\cos\theta - \dot{s}\right)\boldsymbol{i} - a\dot{\theta}\sin\theta\ \boldsymbol{j}$$

M1 **[2]**

Conserving linear momentum horizontally, **M1**

$$m\left(a\dot{\theta}\cos\theta - \dot{s}\right) - M\dot{s} = 0$$

A1

Thus $ma\dot{\theta}\cos\theta = (m+M)\dot{s}$, and so $\dot{s} = \frac{m}{m+M}a\dot{\theta}\cos\theta = \left(1 - \frac{M}{m+M}\right)a\dot{\theta}\cos\theta$

$$\dot{s} = (1-k)a\dot{\theta}\cos\theta$$

M1 **[3]**

Hence,

$$\dot{\boldsymbol{r}} = \left(a\dot{\theta}\cos\theta - (1-k)a\dot{\theta}\cos\theta\right)\boldsymbol{i} - a\dot{\theta}\sin\theta\ \boldsymbol{j}$$

$$= a\dot{\theta}(k\cos\theta\,\boldsymbol{i} - \sin\theta\ \boldsymbol{j})$$

M1 **[1]**

(ii) Conserving energy, **M1**

$$mga = mga\cos\theta + \frac{1}{2}M\left[(1-k)a\dot{\theta}\cos\theta\right]^2 + \frac{1}{2}m\left(a\dot{\theta}\right)^2[(k\cos\theta)^2 + \sin^2\theta]$$

A1

So

$$2mg(1-\cos\theta) = a\dot{\theta}^2(M(1-k)^2\cos^2\theta + mk^2\cos^2\theta + m\sin^2\theta)$$

That is

$$2g(1-\cos\theta) = a\dot{\theta}^2\left(\frac{M}{m}(1-k)^2\cos^2\theta + k^2\cos^2\theta + \sin^2\theta\right)$$

As

$$k = \frac{M}{m+M}$$

$$\frac{m+M}{M} = \frac{1}{k}$$

$$\frac{m}{M} = \frac{1}{k} - 1 = \frac{1-k}{k}$$

and so **M1**

$$\frac{M}{m} = \frac{k}{1-k}$$

$$2g(1-\cos\theta) = a\dot{\theta}^2\left(\frac{k}{1-k}(1-k)^2\cos^2\theta + k^2\cos^2\theta + \sin^2\theta\right)$$

M1

$$= a\dot{\theta}^2(k(1-k)\cos^2\theta + k^2\cos^2\theta + \sin^2\theta)$$

$$= a\dot{\theta}^2(k\cos^2\theta + \sin^2\theta)$$

A1* **[5]**

(iii)

$$\ddot{\boldsymbol{r}} = a\ddot{\theta}(k\cos\theta\,\boldsymbol{i} - \sin\theta\ \boldsymbol{j}) + a\dot{\theta}(-k\dot{\theta}\sin\theta\,\boldsymbol{i} - \dot{\theta}\cos\theta\ \boldsymbol{j})$$

M1

$$= a\left[k(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta)\boldsymbol{i} - (\ddot{\theta}\ \sin\theta + \dot{\theta}^2\cos\theta)\boldsymbol{j}\right]$$

A1

When the particle loses contact with the sphere, $\ddot{\boldsymbol{r}} = -g\boldsymbol{j}$ **E1**

So, $\ddot{\boldsymbol{r}}.(\sin\theta\,\boldsymbol{i} + k\cos\theta\,\boldsymbol{j}) = -g\boldsymbol{j}\,.(\sin\theta\,\boldsymbol{i} + k\cos\theta\,\boldsymbol{j})$ with $\theta = \alpha$ **M1**

Therefore, $a\left[k\ddot{\theta}\cos\alpha\sin\alpha - k\dot{\theta}^2\sin^2\alpha - k\ddot{\theta}\cos\alpha\sin\alpha - k\,\dot{\theta}^2\cos^2\alpha\right] = -kg\cos\alpha$ **M1**

which simplifies to $a\dot{\theta}^2 = g\cos\alpha$ **A1*** **[6]**

Substituting in the final result of (ii),

$$g\cos\alpha\ \ (k\cos^2\alpha + \sin^2\alpha) = 2g(1-\cos\alpha)$$

Thus **M1**

$$(k-1)\cos^3\alpha + 3\cos\alpha - 2 = 0$$

A1

$$3\cos\alpha - 2 = (1-k)\cos^3\alpha$$

$k < 1$, and $\cos\alpha > 0$ so $3\cos\alpha - 2 > 0$ and so $\cos\alpha > \frac{2}{3}$ **E1** **[3]**

10. (i) As the spheres are smooth, as is the table, the momentum of P perpendicular to the direction of the line of centres is unchanged so

$$mu \sin \alpha = mv \sin(\alpha + \theta)$$

M1

and so

$$u \sin \alpha = v \sin(\alpha + \theta)$$

as required. **A1*** **[2]**

Conserving momentum in the direction of the line of centres,

$$mu \cos \alpha = mv \cos(\alpha + \theta) + mw$$

M1 A1

and so

$$u \cos \alpha = v \cos(\alpha + \theta) + w$$

Eliminating u,

$$v \sin(\alpha + \theta) \cos \alpha = v \cos(\alpha + \theta) \sin \alpha + w \sin \alpha$$

M1

So

$$v \sin(\alpha + \theta) \cos \alpha - v \cos(\alpha + \theta) \sin \alpha = w \sin \alpha$$

That is

$$v \sin \theta = w \sin \alpha$$

$$w = v \frac{\sin \theta}{\sin \alpha}$$

M1 A1 **[5]**

[Alternatively, this result can be obtained by conserving momentum perpendicular to the original direction of motion of P.

$$0 = mw \sin \alpha - mv \sin \theta$$ **M2 A1**

(ii) Newton's experimental law of impact in the direction of the line of centres gives

$$w - v \cos(\alpha + \theta) = eu \cos \alpha$$

M1 A1

Substituting for w and u in terms of v gives

$$v \frac{\sin \theta}{\sin \alpha} - v \cos(\alpha + \theta) = \frac{ev \sin(\alpha + \theta) \cos \alpha}{\sin \alpha}$$

M1

Thus

$$\sin \theta - \cos(\alpha + \theta) \sin \alpha = e \sin(\alpha + \theta) \cos \alpha$$

or that is

$$\sin\theta = \cos(\alpha+\theta)\sin\alpha + e\sin(\alpha+\theta)\cos\alpha$$

A1* **[4]**

Expanding and dividing through by $\cos\theta$

$$\tan\theta = \cos\alpha\sin\alpha - \sin^2\alpha\tan\theta + e\sin\alpha\cos\alpha + e\cos^2\alpha\tan\theta$$

M1

$$\tan\theta\,(1+\sin^2\alpha - e\cos^2\alpha) = (1+e)\sin\alpha\cos\alpha$$

Thus, dividing by $\cos^2\alpha$ **M1**

$$\tan\theta\,(\sec^2\alpha + \tan^2\alpha - e) = (1+e)\tan\alpha$$

$$\tan\theta = \frac{(1+e)\tan\alpha}{1+2\tan^2\alpha - e}$$

A1 **[3]**

Let $t = \tan\alpha$,

$$\tan\theta = \frac{(1+e)t}{1-e+2t^2}$$

$$\frac{d(\tan\theta)}{dt} = \frac{(1-e+2t^2)(1+e) - 4t(1+e)t}{(1-e+2t^2)^2}$$

M1 A1

For a maximum,

$$\frac{d(\tan\theta)}{dt} = 0$$

$$(1-e+2t^2) - 4t^2 = 0$$

M1

$$t = \sqrt{\frac{1-e}{2}}$$

A1

$$\tan\theta = \frac{(1+e)}{2(1-e)}\sqrt{\frac{1-e}{2}} = \frac{\sqrt{2}(1+e)}{4\sqrt{1-e}}$$

A1 ft

This is the only stationary value and must be a maximum as for $t = 0$, $\tan\theta = 0$ and as $t \to \infty$, $\tan\theta \to 0$, and for all t , $\tan\theta > 0$. **E1** **[6]**

11. (i) If X is the number of customers who take sand,

$$P(X = r) = \sum_{i=r}^{\infty} \frac{\lambda^i e^{-\lambda}}{i!} \times \binom{i}{r} p^r (1-p)^{i-r}$$

M1 A1 A1

$$= \frac{(\lambda p)^r e^{-\lambda}}{r!} \sum_{I=0}^{\infty} \frac{[\lambda(1-p)]^i}{i!}$$

M1

$$= \frac{(\lambda p)^r e^{-\lambda}}{r!} e^{\lambda(1-p)}$$

$$= \frac{(\lambda p)^r e^{-\lambda p}}{r!}$$

A1*

and so it follows a Poisson distribution with mean λp . **[5]**

(ii)

$$E(mass\ taken)$$

$$= 0.e^{-\lambda p} + kS.\frac{(\lambda p)^1 e^{-\lambda p}}{1!} + [kS + k(1-k)S].\frac{(\lambda p)^2 e^{-\lambda p}}{2!}$$

$$+ [kS + k(1-k)S + k(1-k)^2 S].\frac{(\lambda p)^3 e^{-\lambda p}}{3!} + \cdots$$

M1 A1

$$= e^{-\lambda p} kS \left[\lambda p + \frac{1-(1-k)^2}{1-(1-k)} \frac{(\lambda p)^2}{2!} + \frac{1-(1-k)^3}{1-(1-k)} \frac{(\lambda p)^3}{3!} + \cdots \right]$$

for $k \neq 0$ **E1**

$$= e^{-\lambda p} \frac{kS}{k} \left[\left(\lambda p + \frac{(\lambda p)^2}{2!} + \frac{(\lambda p)^3}{3!} + \cdots \right) - \left((1-k)\lambda p + \frac{(1-k)^2 (\lambda p)^2}{2!} + \frac{(1-k)^3 (\lambda p)^3}{3!} + \cdots \right) \right]$$

$$= e^{-\lambda p} S \left(\left(e^{\lambda p} - 1 \right) - \left(e^{(1-k)\lambda p} - 1 \right) \right)$$

M1

$$= S\left(1 - e^{-k\lambda p}\right)$$

A1*

If $k = 0$, then the expected sand taken is zero which is $S\left(1 - e^{-k\lambda p}\right)$ **E1** **[6]**

(iii) Using the working form part (i), the probability that r customers take sand is

$$\frac{(\lambda p)^r e^{-\lambda p}}{r!}$$

and from (ii) the mass of sand taken is

$$kS\frac{1-(1-k)^r}{1-(1-k)} = S(1-(1-k)^r)$$

so the amount that the merchant's assistant takes is $S(1-k)^r$.

The probability that the assistant takes the golden grain in this case is thus

$$\frac{kS(1-k)^r}{S} = k(1-k)^r$$

M1 A1

The probability that the assistant takes the golden grain is

$$P = \sum_{r=0}^{\infty} k(1-k)^r \frac{(\lambda p)^r e^{-\lambda p}}{r!} = ke^{-\lambda p}\sum_{r=0}^{\infty}\frac{[(1-k)\lambda p]^r}{r!} = ke^{-\lambda p}e^{(1-k)\lambda p} = ke^{-k\lambda p}$$

M1 A1 **[4]**

If $k = 0$, no sand is taken by anyone including the assistant so P should be zero, $0e^0 = 0$ **E1**

As $k \to 1$, $ke^{-k\lambda p} \to e^{-\lambda p}$ which is the probability that no customer takes sand, in the limit, the only way the assistant can take sand, in which case the grain is bound to be in his sand.. **E1** **[2]**

$$P = ke^{-k\lambda p}$$

$$\frac{dP}{dk} = e^{-k\lambda p} - k\lambda pe^{-k\lambda p} = (1-k\lambda p)e^{-k\lambda p}$$

M1

For a maximum, $\frac{dP}{dk} = 0$, so $k = \frac{1}{\lambda p}$. (Note $p\lambda > 1$, so $\frac{1}{\lambda p} < 1$ as required because $k < 1$) **A1**

Justifying that this gives a maximum either:-

$k < \frac{1}{\lambda p} \Rightarrow \frac{dP}{dk} > 0$ and $k > \frac{1}{\lambda p} \Rightarrow \frac{dP}{dk} < 0$ as required

or

$$\frac{d^2P}{dk^2} = -\lambda p(1-k\lambda p)e^{-k\lambda p} - \lambda pe^{-k\lambda p} = \lambda pe^{-k\lambda p}[k\lambda p - 2]$$

and for $k = \frac{1}{\lambda p}$, $\frac{d^2P}{dk^2} = -\lambda pe^{-1} < 0$ as required. **E1** **[3]**

12. For each subset, each integer can be in it or not. Hence the number of possibilities is $2 \times 2 \times 2 \ldots \times 2 = 2^n$. **E1** **[1]**

(i) $P(1 \in A_1) = \frac{1}{2}$ **B1** **[1]**

(ii) $P(t \in A_1 \cap A_2) = \frac{1}{4}$ **B1**

so $P(t \notin A_1 \cap A_2) = \frac{3}{4}$ **M1**

For $A_1 \cap A_2 = \emptyset$, no element is in the intersection, so $P(\, A_1 \cap A_2 = \emptyset\,) = \left(\frac{3}{4}\right)^n$ **M1**

$$P(\, A_1 \cap A_2 \cap A_3 = \emptyset\,) = \left(\frac{7}{8}\right)^n$$

M1 A1 (or B2)

$$P(\, A_1 \cap A_2 \cap \ldots \cap A_m = \emptyset\,) = \left(1 - \frac{1}{2^m}\right)^n$$

M1 A1 (or B2) [7]

(iii) $A_1 \subseteq A_2 \Rightarrow t \in A_1 \cap A_2 \,, t \in A_1' \cap A_2' \,, or, \; t \in A_1' \cap A_2$

M1

So $P(A_1 \subseteq A_2) = \left(\frac{3}{4}\right)^n$ **M1A1** **[3]**

$A_1 \subseteq A_2 \subseteq A_3 \Rightarrow t \in A_1 \cap A_2 \cap A_3 \,, t \in A_1{}' \cap A_2 \cap A_3, t \in A_1{}' \cap A_2{}' \cap A_3 \; or \,, t \in A_1{}' \cap A_2{}' \cap A_3{}'$

M1A1

So $P(A_1 \subseteq A_2 \subseteq A_3) = \left(\frac{4}{8}\right)^n = \left(\frac{1}{2}\right)^n$ **M1A1** **[4]**

$A_1 \subseteq A_2 \ldots A_m \Rightarrow t \in A_1 \cap A_2 \ldots A_m \,, t \in A_1' \cap A_2,,, A_m, t \in A_1' \cap A_2' \ldots A_m \ldots \;, t \in A_1' \cap A_2' \ldots A_m{}'$

M1A1

$$P(A_1 \subseteq A_2 \subseteq \cdots \subseteq A_m) = \left(\frac{m+1}{2^m}\right)^n$$

M1A1 **[4]**